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LETTER TO THE EDITOR

Failure of perturbation theory in random field models

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Abstract. The failure of perturbation theory is demonstrated in a toy model which describes a domain wall in a random field Ising ferromagnet. It is argued that there are non-perturbative terms which are related to rare random field distributions for which the Hamiltonian have many minima while the perturbative solutions are unique.

For those who are good at mathematics, there is now a proof available that the three-dimensional random field Ising model (for a review see [1]) shows order for sufficiently weak random fields at $T=0$ [2] or at low temperature [3]. This is in agreement with simple arguments [4], but disagrees with various sophisticated, though approximate, calculations [5, 6] based on standard many-body methods. The reason for the failure is not yet fully understood and the forthcoming discussion aims to provide a better understanding.

In this letter we want to discuss the difficulties which arise in the perturbation-theoretical treatment of a domain wall in the random field Ising model. If d is the space dimension, the domain wall is roughly parallel to the hyperplane P of the first $(d-1)$ coordinates and z_i denotes the height of the wall in the dth

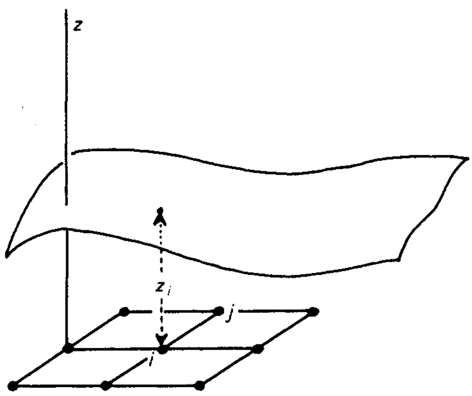

Figure 1. A domain wall for $d=3$ dimensions.

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direction (see figure 1) above the point $i \in P$, then the Hamiltonian is

$$\mathcal{H} = \tfrac{1}{2}g \sum_i z_i^2 + \tfrac{1}{2}J \sum_{\langle ij \rangle} (z_i - z_j)^2 + \sum_i \int_{-\infty}^{\infty} \mathrm{d}\zeta \, H_i(\zeta) f(\zeta - z_i) \tag{1}$$

where g and J are constants, $\langle ij \rangle$ denotes pairs of nearest neighbours on a $(d-1)$-dimensional simple cubic lattice, and the $H_i(\zeta)$ are independent, random, frozen variables:

$$\overline{H_i(\zeta) H_j(\zeta')} = H^2 \delta_{ij} \delta(\zeta - \zeta') \tag{2}$$

where H is a real constant.

The function $f(z)$ should satisfy the condition $f(z) \simeq z/|z|$ for $|z|$ larger than the wall thickness, so that $f(z)$ represents Ising spins on both sites of the wall. However, we want $f(z)$ to be a smooth function in order to be able to apply perturbation theory. Thus, we assume $f'(z) \simeq f'(0)$ for $|z| < 1/f'(0)$, while $f'(z)$ vanishes rapidly for $|z| > 1/f'(0)$. Since the radius of convergence of f is important, it is sometimes appropriate to use a special form of f. We choose

$$f(z) = \tanh z. \tag{3}$$

This system has been studied by various authors [7-9]. In the limit $g \to 0$, $T \to 0$ and $H \ll J$ it has been argued [1, 7-9] (and we believe this is right) that for a wall of size L^{d-1},

$$\overline{\langle z^2 \rangle} \equiv \frac{1}{L^{d-1}} \sum_i \overline{\langle z_i^2 \rangle} \sim (H/J)^{4/3} L^{(5-d)/3}. \tag{4}$$

The brackets $\langle \ldots \rangle$ denote a thermal average and the bar an average on the field distributions.

On the other hand, if the last term of (1) is treated as a perturbation, it can be shown [10] that all terms of the perturbation expansion converge for $d > 3$ and the result is

$$\overline{\langle z^2 \rangle} \sim (H/J)^2 L^{(5-d)/2} \tag{5}$$

with a temperature-dependent prefactor.

Since neither (4) nor (5) can be exactly derived, it is appropriate to investigate a model where exact results can be obtained [8, 11, 12]. This model is the one-dimensional version of (1). i takes a single value 1, so that the second term of (1) vanishes and, writing $z_i = z$, (1) becomes

$$\mathcal{H} = \tfrac{1}{2}gz^2 + \int_{-\infty}^{\infty} \mathrm{d}\zeta \, H(\zeta) f(\zeta - z). \tag{6}$$

In the case where $f(z) = z/|z|$ it can be exactly proved that [8, 12]

$$\overline{\langle z^2 \rangle} \simeq (H/g)^{4/3} \tag{7}$$

and this should obviously be correct for model (3) as well if $\langle z^2 \rangle$ is much larger than the squared wall thickness (1), i.e. if $H \gg g$. On the other hand, for $H \ll g$, one would like to apply perturbation theory. It starts with the Taylor expansion of (6), namely

$$\mathcal{H} = \tfrac{1}{2}gz^2 + \sum_{n=0}^{\infty} z^n Q_n / n! \tag{8}$$

where

$$Q_n = (-1)^n \int_{-\infty}^{\infty} \mathrm{d}\zeta f^{(n)}(\zeta) H(\zeta) \tag{9}$$

$f^{(n)}$ denotes the nth derivative of f. The expansion (8) converges, with the choice (3), if

$$|z| < \pi/2. \tag{10}$$

The minimum z_0 of (8) satisfies

$$\mathrm{d}\mathcal{H}/\mathrm{d}z \equiv (g+Q_2)z_0 + Q_1 + \tfrac{1}{2}Q_3 z_0^2 + \tfrac{1}{6}Q_4 z_0^3 + \ldots. \tag{11}$$

Apart from exceptional cases, to be addressed below, all Q_n are of order H. Hence, if $H \ll g$, (11) generally has a single solution:

$$z_0 \simeq \left(Q_1 + \frac{Q_3 Q_1^2}{2g^2}\right)(g+Q_2)^{-1} \simeq \frac{Q_1}{g} - \frac{Q_1 Q_2}{g^2} + \frac{Q_1 Q_2^2}{g^3} + \frac{Q_1^2 Q_3}{2g^3}. \tag{12}$$

With a high probability, Q_1, Q_2, Q_3 are of order $H \ll g$, and (12) satisfies (10). Equation (12) yields

$$\overline{\langle z^2\rangle_{T=0}} = \overline{\langle z_0^2\rangle} = \overline{Q_1^2}/g^2 + 3\overline{Q_1^2 Q_2^2}/g^4 + \overline{Q_1^3 Q_3}/g^4 + \mathrm{O}(H^6/g^6)$$

or, using (9) and (2),

$$\begin{aligned}\overline{\langle z^2\rangle_{T=0}} &= \frac{H^2}{g^2} V_1 + \frac{3H^4}{g^4} V_1 \int_{-\infty}^{\infty} \left[f''^2(\zeta) + f'(\zeta) f'''(\zeta) \right] \mathrm{d}\zeta \\ &\quad + \frac{6H^4}{g^4} \left(\int_{-\infty}^{\infty} f'(\zeta) f''(\zeta)\, \mathrm{d}\zeta \right)^2 + \mathrm{O}\left(\frac{H^6}{g^6}\right) \\ &= (H/g)^2 V_1 + \mathrm{O}(H^6/g^6)\end{aligned} \tag{13}$$

where

$$V_1 = \int_{-\infty}^{\infty} f'^2(\zeta)\, \mathrm{d}\zeta. \tag{14}$$

Thus we have been surprised to see that the term in $(H/g)^4$ vanishes! Using the replica trick, it can be shown that all terms vanish beyond second order. A similar result was already obtained by Engel [11] for the iterative (instead of perturbative) solution in the case of a slightly different model. Thus, the result (7) cannot be obtained by perturbation theory, even summing all terms (since they all vanish!). We are left with two possibilities. The first one is that, at $T=0$, $\overline{\langle z^2\rangle} = (H/g)^2 V_1$ for H/g smaller than some threshold. The second one is

$$\overline{\langle z^2\rangle_{T=0}} = V_1 H^2/g^2 + \text{non-perturbative terms}. \tag{15}$$

It will now be argued that this second possibility is the most plausible one. Indeed, there is some probability that the random field distribution is such that (12) does not satisfy (10), and is therefore not reliable. This happens for instance if $|Q_1| > g$ or $Q_2 < -g$.

The probabilities that $|Q_1| > g$ and that $Q_2 < -g$ are not independent. Indeed, if $Q_1 \gg g$, then Q_2 has a probability to be smaller than $-g$ and conversely. Using (9) and (3), the probability that $|Q_1| > g$ can be roughly evaluated by replacing $f'(\zeta)$ by $f'(0)$

for $|z|<1/f'(0)$ and 0 for $|z|>1/f'(0)$. Replacing the integral (9) for $n=1$ by its Riemann definition, one divides the interval $[-1/f'(0), 1/f'(0)]$ in small intervals of length a and (9) is approximated by

$$Q_1 = \sum_{n=-1/f'(0)a}^{1/f'(0)a} H\sqrt{a}\sigma_n f'(0) \tag{16}$$

where $\sigma_n = \pm 1$. The factor $\sqrt{a}$ is introduced in order to have

$$\overline{\left(\sum_{n=x}^{x+\mathrm{d}x/a} H\sqrt{a}\sigma_n\right)^2} = H^2\,\mathrm{d}x$$

independently of a. Now, the probability that (16) is larger than g is

$$P = \text{constant} \times (H\sqrt{f'(0)}/g)^\alpha \exp(-Kg^2/H^2 f'(0)) \tag{17}$$

where $\alpha = 1$ and $K=\frac{1}{2}$, but the approximation (16) is not precise enough for these evaluations to be reliable. However, the general form (17) is probably correct.

If Q_1 and Q_2 are of order g, H can have several minima and then formula (12) is unreliable. Thus, our speculation is

$$\overline{\langle z^2\rangle}_{T=0} = V_1 H^2/g^2 + \text{constant} \times (H\sqrt{f'(0)}/g)^\gamma \times \exp(-Kg^2/H^2 f'(0)) \qquad H \ll g/\sqrt{f'(0)}. \tag{18}$$

We shall now consider another quantity where such a non-perturbative expression can be derived somewhat less speculatively. This quantity is the fourth moment $\langle \delta z^4\rangle$

$$\delta z = z - z_0$$

which we want to evaluate at low, but non-vanishing, temperature.

The perturbation-theoretic treatment will first be outlined. The length unit will be chosen such that $f'(0)=1$. Assuming all $Q_n \approx H \ll g$, (8) becomes

$$\delta\mathcal{H} = \tfrac{1}{2}g'\delta z^2 + \tfrac{1}{6}Q_3\delta z^3 + \tfrac{1}{24}Q_4(\delta z^4 + 4z_0\delta z^3) + \ldots$$

where $\delta\mathcal{H} = \mathcal{H}(z) - \mathcal{H}(z_0)$ and

$$g' = g + Q_2 + Q_3 z_0 + \tfrac{1}{2}Q_4 z_0^2 + \ldots.$$

A straightforward calculation yields

$$\langle \delta z^4\rangle = 3(T^2/g'^2)(1 + AQ_4 T/g'^2 + \ldots)$$

where A is a numerical coefficient. Thus, in the low-temperature limit, for almost all field configurations,

$$\langle \delta z^4\rangle \simeq 3T^2/g^2. \tag{19}$$

It will now be argued non-perturbatively that, for fixed H and g, the correct low-temperature behaviour is $\langle \delta z^4\rangle \sim T$, so therefore much larger than (19). To obtain this result, it is necessary to consider the rare field configurations for which $Q_2 < -g$ and/or $|Q_1| > g$. For those configurations it is not possible to consider the field much smaller than g. On the contrary, its average between -1 and 1 is of order Q_1, i.e. of order g since $|Q_1| > g$ (configurations with $|Q_1| \gg g$ can be neglected because the probability analogous to (17) would be much smaller). So, the contribution to $\langle \delta z^4\rangle$ of field configurations with large Q_1 is obtained as follows: first calculate $\langle \delta z^4\rangle$ for $H \approx g$, then multiply by (17). For $H \approx g$, $\langle \delta z^4\rangle$ can be roughly evaluated as in the case $H \gg g$ treated in [10]. Thus, we first recall that case.

If $H \gg g$, the main contribution to higher moments $\overline{\langle \delta z^{2p} \rangle}$ arises from rare field configurations where $\mathscr{H}(\lambda)$ has two minima z_1, z_2 at large 'horizontal' distance $|z_1 - z_2| \approx \overline{\langle z^2 \rangle}^{1/2}$ and small 'vertical' distance $|\mathscr{H}(z_1) - \mathscr{H}(z_2)| < T$. The probability that this occurs is of order $T/g\langle z^2 \rangle$ and it follows

$$\overline{\langle \delta z^{2p} \rangle} \approx (T/g\overline{\langle z^2 \rangle})\overline{\langle z^2 \rangle}^p \approx (T/g)(H/g)^{4(p-1)/3}.$$

For $H \approx g$ this is of order T/g. As was said above, the contribution to $\overline{\langle \delta z^4 \rangle}$ of field configurations with $Q_1 \approx g$ is obtained by multiplication by (17). Then one should add the contribution (19) of field configurations with $Q_1 \approx H$. The result is

$$\overline{\langle \delta z^4 \rangle} \simeq 3T^2/g^2 + \text{constant} \times (T/g)(H/g)^\alpha \exp(-Kg^2/H^2). \qquad (20)$$

In the low-temperature limit the second term dominates for any fixed H and g. On the other hand, this term cannot be obtained by perturbation theory since $\exp(-Kg^2/H^2)$ has no expansion in powers of H/g. Perturbation theory will give only the first term of (20).

As in similar cases [11, 13, 14] a standard many-body technique (here perturbation theory) is seen to fail because the Hamiltonian (8) has multiple minima. If perturbation theory fails in one dimension, it has no reason to succeed in more dimensions.

Appendix 1. Proof of (15)

(It will not be proved that non-perturbative terms are non-zero!)

The replicated Hamiltonian can be written, apart from an additive constant, as

$$\tilde{\mathscr{H}} = \tfrac{1}{2}g \sum_{\alpha=1}^{n} z_\alpha^2 + \tfrac{1}{4}\beta H^2 \sum_{\alpha,\gamma=1}^{n} V(z_\alpha - z_\beta) \qquad (\text{A}1)$$

where α, γ are the replica indices, n is the number of replicas and

$$V(z) = \int_{-\infty}^{\infty} \mathrm{d}\zeta\, [f(\zeta - z) - f(\zeta)]^2$$
$$\simeq V_1 z^2 + V_2 z^4 + \dots$$

where

$$V_1 = \int_{-\infty}^{\infty} \mathrm{d}\zeta\, f'^2(\zeta)$$

$$V_2 = \int_{-\infty}^{\infty} \mathrm{d}\zeta\, [\tfrac{1}{4}f''^2(\zeta) + \tfrac{1}{3}f'(\zeta)f'''(\zeta)]$$
$$= -\tfrac{1}{12}\int_{-\infty}^{\infty} \mathrm{d}\zeta\, f''^2(\zeta).$$

The Hamiltonian (A1) is

$$\tilde{\mathscr{H}} = \tilde{\mathscr{H}}_0 + \tilde{\mathscr{H}}_4 + \tilde{\mathscr{H}}_6 + \dots$$

with

$$\tilde{\mathscr{H}}_{2p} = \frac{H^2}{4T} V_p \sum_{\alpha\gamma} (z_\alpha - z_\gamma)^{2p}.$$

The unperturbed Hamiltonian $\tilde{\mathscr{H}}_0$ has one eigenvector $|1\rangle$ with eigenvalue g and $(n-1)$ eigenvectors $|2\rangle, \dots, |n\rangle$ with eigenvalue $(g + n\varepsilon)$ with $\varepsilon = H^2V_1/4T$. Let

$Z_1, Z_2, \ldots, Z_n$ be the components of $(z_1, \ldots, z_N, \ldots, z_n)$ along the eigenvectors. The perturbation $(\tilde{\mathscr{H}}_4+\tilde{\mathscr{H}}_6+\ldots)$ does not depend on Z_1. Therefore $\langle Z_1^2\rangle = \langle Z_1^2\rangle_o$, the unperturbed average value. On the other hand

$$\langle Z_2^2\rangle = \frac{\sum_{p=0}^{\infty}(1/p!)\langle Z_2^2(\tilde{\mathscr{H}}_4+\tilde{\mathscr{H}}_6+\ldots)^p\rangle_o(-\beta)^p}{\sum_{p=0}^{\infty}(1/p!)\langle(\tilde{\mathscr{H}}_4+\tilde{\mathscr{H}}_6+\ldots)^p\rangle_o(-\beta)^p}.$$

Let first $\tilde{\mathscr{H}}_6, \tilde{\mathscr{H}}_8, \ldots,$ be neglected. The term of order p in the denominator is equal to a function of n multiplied by

$$(H^2/T^2)^p\langle Z_2\rangle_o^{4p} = \left(\frac{H^2}{T^2}\right)^p\left(\frac{T}{g+n\varepsilon}\right)^{2p} = \left(\frac{H}{g+n\varepsilon}\right)^{2p}.$$

The term of order p in the numerator is equal to the same thing times $\langle Z_2^2\rangle_o$.

Now, what is the effect of $\tilde{\mathscr{H}}_6$? For instance a term in the denominator with one factor $\tilde{\mathscr{H}}_4$ replaced by $\tilde{\mathscr{H}}_6$ is

$$\beta^p\langle\mathscr{H}_4^{p-1}\mathscr{H}_6\rangle_o = \left(\frac{H}{g+n\varepsilon}\right)^{2p}\frac{T}{g+n\varepsilon}\times \text{a function of } n.$$

The general term of $\langle Z_2^2\rangle$ is easily seen to have the form

$$A_{pq}(n)\left(\frac{H}{g+n\varepsilon}\right)^{2p}\left(\frac{T}{g+n\varepsilon}\right)^q\langle Z_2^2\rangle_o.$$

It follows that

$$\overline{\langle z^2\rangle} = \frac{1}{n}\sum_\alpha\langle z_\alpha^2\rangle = \frac{1}{n}[\langle Z_1^2\rangle + (n-1)\langle Z_2^2\rangle]$$
$$= \frac{1}{n}\sum_\alpha\langle z_\alpha^2\rangle_o^2 + \frac{n-1}{n}\frac{T}{g+n\varepsilon}\sum_{p=1}^{\infty}\sum_{q=0}^{\infty}A_{pq}(n)\left(\frac{H}{g+n\varepsilon}\right)^{2p}\left(\frac{T}{g+n\varepsilon}\right)^q.$$

The limit $n=0$ should be taken. Since the result should be finite, all $A_{pq}(n)$ should contain a factor n for $p>0$. This is proved for A_{10} in appendix 2. At $T=0$, the above expansion reduces to its first term and this proves (15).

Appendix 2. Calculation of $A_{10}(n)$

One can choose $Z_2 = (z_1 - z_2)/\sqrt{2}$. Then

$$\langle Z_2^2\rangle = \frac{\langle Z_2^2(1-\beta\tilde{\mathscr{H}}_4+\ldots)\rangle_o}{\langle 1-\beta\tilde{\mathscr{H}}_4+\ldots\rangle_o}$$
$$= \langle Z_2^2\rangle_o - \beta[\langle Z_2^2\tilde{\mathscr{H}}_4\rangle_o - \langle Z_2^2\rangle_o\langle\tilde{\mathscr{H}}_4\rangle_o] + \ldots$$
$$= \langle Z_2^2\rangle_o - \frac{H^2}{8T^2}V_2\sum_{\alpha\gamma}[\langle(z_1-z_2)^2(z_\alpha-z_\gamma)^4\rangle_o - \langle(z_1-z_2)^2\rangle_o\langle(z_\alpha-z_\gamma)^4\rangle_o] + \ldots$$
$$= \langle Z_2^2\rangle_o - \frac{H^2}{4T^2}V_2[\langle(z_1-z_2)^6\rangle_o - \langle(z_1-z_2)^2\rangle_o\langle(z_1-z_2)^4\rangle_o]$$
$$- \frac{H^2}{2T^2}V_2(n-2)[\langle(z_1-z_2)^2(z_1-z_3)^4\rangle_o - \langle(z_1-z_2)^2\rangle_o\langle(z_1-z_3)^4\rangle_o].$$

The calculation is easier if one chooses a new basis with $z_1 - z_3 = Z_2\sqrt{2}$ and $Z_3 = (z_1 + z_2 - 2Z_2)/\sqrt{6}$. Hence $2(z_1 - z_2) = Z_3\sqrt{6} + Z_2\sqrt{2}$. It follows that

$$\begin{aligned}
\langle Z_2^2\rangle - \langle Z_2^2\rangle_o &= -\frac{H^2}{4T^2} V_2[8\langle Z_2^6\rangle_o - 8\langle Z_2^2\rangle_o\langle Z_2^4\rangle_o] \\
&\quad -\frac{H^2}{2T^2} V_2(n-2)[\langle 4Z_2^4(Z_3\sqrt{\tfrac{3}{2}} + Z_2/\sqrt{2})^2\rangle_o - 4\langle Z_2^4\rangle_o\langle (Z_3\sqrt{\tfrac{3}{2}} + Z_2\sqrt{2})^2\rangle_o] \\
&= -\frac{2H^2}{T^2} V_2[\langle Z_2^6\rangle_o - \langle Z_2^2\rangle_o\langle Z_2^4\rangle_o] - \frac{H^2}{T^2} V_2(n-2)[\langle Z_2^6\rangle_o - \langle Z_2^2\rangle_o\langle Z_2^4\rangle_o] \\
&= -\frac{H^2}{T^2} V_2 n[\langle Z_2^6\rangle_o - \langle Z_2^2\rangle_o\langle Z_2^4\rangle_o] \\
&= -12(H^2/T^2) V_2 n\langle Z_2^2\rangle_o^3.
\end{aligned}$$

Hence

$$A_{10}(n) = -12 V_2 n.$$

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